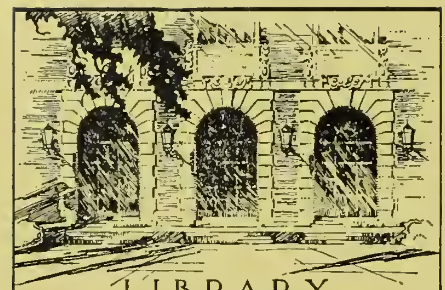

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ON SHOCKS ATTACHED TO A WALL IN A STATIONARY OR PSEUDO-STATIONARY TWO-DIMENSIONAL FLOW

by

Y. S. Chow

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ON SHOCKS ATTACHED TO A WALL IN A STATIONARY OR PSEUDO-STATIONARY TWO-DIMENSIONAL FLOW

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Y. S. Chow

1. Introduction

Shock waves attached to a solid boundary have been discussed by many authors in stationary or pseudo-stationary two-dimensional flows. In the stationary case, Tsien [9] showed that at the attachment point, 0, of a normal shock, the Mach numbers before and after the shock are determined by the ratio of the curvatures of the boundary. He assumed that the flow ahead of the shock is isentropic and irrotational and that all the quantities involved are finite and continuous except in crossing the shock. Lin and Rubinov [3] extended his result by allowing the curvature of the shock and the pressure gradient behind the shock to become infinite at the point 0. Let $M_c^2 = \frac{\gamma+1}{2} + \frac{1}{2}\sqrt{(\gamma+1)^2 + 4}$. They proved that over a convex (concave) boundary with continuous curvature a normal shock can be formed only when the Mach number ahead of the shock is above (below) M_c. By assuming finiteness of the curvatures of the boundary before and after the shock, they found that the normal shock in a neighborhood of the point 0 is approximately $x = \beta y^{3/2}$, where β is a constant, $y = 0$ is the tangent of the boundary, $x = 0$ is that of the shock, and 0 is the origin. At a slightly earlier time than Tsien's paper, Thomas [5] obtained a relation between the curvatures of the stream lines behind the shock and those of the shock, and later he [6] extended the result to the derivatives of the curvatures, if the flow is stationary and the flow ahead of the shock is uniform. Taub [4] gave a simpler derivation of Thomas' results and extended them to pseudo-stationary flows.

Brown [1], using devices due to Thomas, extended Lin and Rubinov's results to stationary, iso-energetic flows. The author extends Taub's treatment to obtain a general relation among the curvatures of the streak lines behind the shock, before the shock, and of the shock, for a general stationary or pseudo-stationary flow.

Let the boundary be a straight wall and let the flow ahead of the shock be uniform. In part 4, the author proves that, if the shock is normal to the wall

at 0, the derivatives of odd order of the curvature of the shock are zero at 0, provided that the shock is regular there. In part 5, the curvature of the shock is allowed to become infinite at 0 and approximations for the streak lines and the shock near the point 0 are obtained.

2. Main Equations

(A) Let $\mathcal{L}$ be a shock in a two-dimensional stationary or pseudo-stationary flow of an ideal gas with constant specific heats and without heat conductivity and viscosity, (x_1, x_2) a fixed cartesian coordinate system in the plane, and the shock be described by the parametric equations

$$x_i = (1 - \epsilon) a_i(s) + \epsilon t\, a_i(s) \tag{2.1}$$

where $i = 1, 2$, s is the arc length along the shock, and $\epsilon = 1$ or 0 according to whether the flow is pseudo-stationary or stationary. For simplicity, the summation convention will be used, i.e., in a term repeated indices are dummy indices of the summation from 1 to 2, and all the free indices may take values 1 and 2 except when the contrary is mentioned. We will assume that continuity and differentiability requirements on the functions involved in the discussion of this section are always satisfied.

Choose independent variables (s, T) by

$$x_i(s,T) = (1 - \epsilon) a_i(s,T) + \epsilon t a_i(s,T) , \tag{2.2}$$

where

$$\frac{\partial a_i(s,T)}{\partial T} = \frac{U_i}{V} \quad \text{for } T \neq 0, \; U_i(s,T) = u_i(s,T) - \epsilon a_i(s,T), \tag{2.3}$$

$$V(s,T) = \sqrt{U_i\, U_i} , \qquad a_i(s,0) = a_i(s) \tag{2.4}$$

and $u_i(s,T)$ and $u_i(s, -T)$, for $T > 0$, are the velocity components relative to the fixed coordinate system (x_1, x_2) after the shock and before the shock, respectively.

Define

$$\lambda_i(s,T) = \frac{\partial a_i(s,T)}{\partial s} , \quad \lambda(s,T) = \sqrt{\lambda_i\, \lambda_i} , \quad \tan \alpha(s,T) = \frac{\lambda_2}{\lambda_1} \tag{2.5}$$

$$U_t(s,T) = \lambda_i\, U_i \quad \text{and} \quad U_n(s,T) = \nu_i\, U_i , \tag{2.6}$$

where

$$\mathcal{V}_1(s,T) = \lambda_2 , \qquad \mathcal{V}_2(s,T) = -\lambda_1 \quad . \tag{2.7}$$

Put

$$\tan \mu(s,T) = \frac{U_t}{U_n} \quad \text{and} \quad \tan \zeta_2(s,T) = \frac{U_2}{U_1} \; . \tag{2.8}$$

Then

$$\zeta_2(s,T) = \alpha + \mu - \frac{\pi}{2} \; . \tag{2.9}$$

In the following, partial differentiation with respect to s and T will be denoted by a subscript after a comma, for example,

$$f_{,s} = \frac{\partial f}{\partial s} , \qquad f_{,s^2T} = \frac{\partial^3 f}{\partial^2 s \, \partial T} \; .$$

The curves T = constant in the (a_1,a_2) plane will be called displaced shocks and s = constant streak lines. It is obvious that the shock itself is the curve, $T = 0$, and in the stationary case, streak lines are the stream lines. We will put

$$k(s,T) = \alpha_{,s} \quad \text{and} \quad K(s,T) = \zeta_{2,T} \text{ or } -\zeta_{2,T} \tag{2.10}$$

according to whether $U_1(s,T) \geq 0$ or $U_1(s,T) \leq 0$.

(B) Fundamental equations. The hydrodynamical equations for the flow are

$$\frac{\partial(\rho U_i)}{\partial a_i} + 2\epsilon\rho = 0 , \tag{2.11}$$

$$\rho\left(U_j \frac{\partial U_i}{\partial a_j} + \epsilon U_i\right) + \frac{\partial p}{\partial a_i} = 0 , \tag{2.12}$$

and

$$U_i \frac{\partial S}{\partial a_i} = S_{,T} = 0 , \tag{2.13}$$

where ρ, p and S are the density, pressure and specific entropy, respectively. The Rankine-Hugoniot equations (along the shock) are

$$U_t(s,0) = U_t' \ , \tag{2.14}$$

$$\eta U_n(s,0) = U_n' \ , \tag{2.15}$$

$$y(s,0) = \frac{p}{p'} = 1 + \frac{2\gamma}{\gamma+1}(\sigma^2 - 1) \ , \tag{2.16}$$

and

$$\eta(s,0) = \frac{\rho}{\rho'} = \frac{(\gamma+1)\sigma^2}{2+(\gamma-1)\sigma^2} \tag{2.17}$$

where $f(s,0)$ and $f'(s,0)$ of the variable f are defined by

$$f(s,0) = \lim_{T \to 0+} f(s,T) \ , \qquad f'(s,0) = \lim_{T \to 0-} f(s,T) \ ,$$

γ is the ratio of specific heats, $\sigma(s,0) = \frac{U_n'}{c'} = m' \cos \mu'$, $m(s,T) = \frac{v}{c}$ and $c^2(s,T) = \frac{\gamma p}{\rho}$.

Let

$$\zeta_1(s,T) = \frac{1}{\gamma} \log p, \quad C_1(s,T) = 0, \quad C_2(s,T) = \frac{-\lambda \cos \mu}{v} \ ,$$

and

$$\delta_i(s,T) = \zeta_{i,s} - \epsilon C_i \ .$$

From (2.11) to (2.17), we have [4;(5.10)]

$$\zeta_{i,T}\,(s,T) = A_{ij}\,\delta_j \ , \quad \delta_i(s,T) = a_{ij}\,\zeta_{i,T} \tag{2.18}$$

where

$$\|A_{ij}(s,T)\| = \frac{1}{\lambda(1-m^2 \cos^2 \mu)} \left\| \begin{matrix} \sin \mu & m^2 \cos \mu \\ (1 - \frac{1}{m^2})\cos \mu & \sin \mu \end{matrix} \right\| \ ,$$

$$\|a_{ij}(s,T)\| = \|A_{ij}(s,T)\|^{-1} = \lambda \left\| \begin{matrix} \sin \mu & -m^2 \cos \mu \\ (\frac{1}{m^2} - 1) \cos \mu & \sin \mu \end{matrix} \right\| . \tag{2.19}$$

(C) Derivation of the main equations. From (2.18), along the shock,

$$\zeta_{i,T}(s,0) = A_{ij}\,\delta_j \tag{2.20}$$

$$\zeta_{i,T}'(s,0) = A_{ij}'\,\delta_j' \tag{2.21}$$

By differentiating (2.16),

$$\delta_1(s,0) - \delta_1'(s,0) = \frac{4\,\sigma\,\sigma_{,s}}{2\gamma\sigma^2 + 1 - \gamma} \equiv B_1, \text{ say.} \qquad (2.22)$$

From (2.14) and (2.15),

$$\tan\mu(s,0) = \eta \tan\mu'$$

and then

$$\begin{aligned} \delta_2 \sec^2\mu(s,0) - \eta\,\delta_2'\sec^2\mu'(s,0) &= k(\sec^2\mu - \eta\sec^2\mu') + \eta_{,s}\tan\mu' \\ &\qquad\qquad (2.23) \\ &= k(\sec^2\mu - \eta\sec^2\mu') + \frac{4\tan\mu}{2+(\gamma-1)\sigma^2} \cdot \frac{\sigma_{,s}}{\sigma} \equiv B_2, \text{ say.} \end{aligned}$$

(2.20) to (2.23) give six equations for eight variables $\zeta_{i,T}$, $\zeta'_{i,T}$, δ_i and δ_i'. Therefore we can solve for δ_i in terms of $\zeta'_{i,T}$, k, B_1, B_2 and the flow variables. The matrix of coefficients of (2.20) to (2.23) is

$$\begin{array}{|cccccccc||} \delta_1 & \delta_2 & \zeta_{1,T} & \zeta_{2,T} & \delta_1' & \delta_2' & = & (\zeta'_{i,T}, k, B_1, B_2) \\ -A_{11} & -A_{12} & 1 & 0 & 0 & 0 & & 0 \\ -A_{21} & -A_{22} & 0 & 1 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & A'_{11} & A'_{12} & & \zeta'_{1,T} \\ 0 & 0 & 0 & 0 & A'_{21} & A'_{22} & & \zeta'_{2,T} \\ 1 & 0 & 0 & 0 & -1 & 0 & & B_1 \\ 0 & \sec^2\mu & 0 & 0 & 0 & -\eta\sec^2\mu' & & B_2 \end{array} .$$

Let D be the determinant consisting of the first six columns of this matrix, and D_1 be that obtained by replacing the first column in D by the last column of this matrix. Then

$$D(s,0) = \frac{\sec^2\mu}{1 - m'^2\cos^2\mu'}$$

$$D_1(s,0) = \left(A'_{22}\,\zeta'_{1,T} - A'_{12}\,\zeta'_{2,T} + \frac{B_1}{1 - m'^2\cos^2\mu'}\right)\sec^2\mu .$$

Therefore,

$$\delta_1(s,0) = \sin\mu' \cdot \zeta'_{1,T} - m'\sigma\,\zeta'_{2,T} + B_1 \; . \tag{2.24}$$

Similarly,

$$\delta_2(s,0) = \frac{\eta\cos^2\mu}{\cos^2\mu'}\left(\frac{1-m'^2}{m'^2}\cos\mu' \cdot \zeta'_{2,T}\right) + B_2\cos^2\mu \; . \tag{2.25}$$

Let

$$\phi(s,T) = U_{1,2} - U_{2,1} \; .$$

Then, [4;(4.19)]

$$U_n\phi(s,T) = vv_{,s} + \frac{p_{,s}}{\rho} + \epsilon\, U_t \; . \tag{2.26}$$

Let

$$h(s,T) = \frac{v^2}{2} + \frac{1}{\gamma-1}c^2 \; .$$

Then

$$h_{,s}(s,T) = vv_{,s} + \frac{2c}{\gamma-1}c_{,s} = vv_{,s} + \frac{\gamma}{\gamma-1}\left(\frac{p_{,s}}{\rho} - \frac{p\rho_{,s}}{\rho^2}\right) \tag{2.27}$$

Therefore

$$\left(\frac{v_{,s}}{v} - \frac{c_{,s}}{c}\right)(s,T) = \frac{v_{,s}}{v}\left[1 + \frac{(\gamma-1)v^2}{2c^2}\right] - \frac{\gamma-1}{2c^2}\,h_{,s}$$

$$= \frac{1}{v^2}\left(U_n\,\phi - \frac{p_{,s}}{\rho} - \epsilon\, U_t\right)\left[1 + \frac{(\gamma-1)v^2}{2c^2}\right] - \frac{\gamma-1}{2c^2}\,h_{,s} \; .$$

Hence

$$\left(\frac{v_{,s}}{v} - \frac{c_{,s}}{c}\right)(s,T) = \left(\frac{1}{v^2} + \frac{\gamma-1}{2c^2}\right)U_n\phi - \left(\frac{1}{m^2} + \frac{\gamma-1}{2}\right)\zeta_{1,s} - \frac{\epsilon U_t}{v^2}\;B_3, \tag{2.28}$$

where

$$B_3(s,T) = \frac{\gamma-1}{2c^2}\left(h_{,s} + \epsilon\, U_t\right) \; . \tag{2.29}$$

Now [2; p.10]

$$\frac{\partial p\rho^{-\gamma}}{\partial s}(s,T) = \frac{p\rho^{-\gamma}}{c_v}S_{,s} = p_{,s}\rho^{-\gamma} - \gamma p\rho^{-\gamma-1}\rho_{,s} \; ,$$

where c_v is the specific heat at constant volume. Hence,

$$S_{,s}(s,T) = c_v\left(\frac{p_{,s}}{p} - \frac{\gamma}{\rho}\,\rho_{,s}\right) \; . \tag{2.30}$$

From (2.26) and (2.27),

$$U_n\phi(s,T) = (h_{,s} + \epsilon U_t) - \frac{p}{(\gamma-1)\rho}\left(\frac{p_{,s}}{p} - \frac{\gamma\rho_{,s}}{\rho}\right) .$$

$$= \frac{2c^2}{\gamma-1}(B_3 - \frac{1}{2\gamma c_v} S_{,s}).$$

Hence,

$$B_3(s,T) = \frac{\gamma-1}{2c^2} U_n\phi + \frac{1}{2\gamma c_v} S_{,s} . \tag{2.31}$$

Since $\sigma(s,0) = m' \cos \mu'$, we have

$$\sigma_{,s}(s,0) = m'_{,s} \cos \mu' - m'\mu'_{,s} \sin \mu' = \left(\frac{v'_{,s}}{v'} - \frac{c'_{,s}}{c'}\right)\sigma - (\zeta'_{2,s} - k)m' \sin \mu' .$$

By (2.28) and (2.31),

$$\sigma_{,s}(s,0) = -\left(\frac{1}{m'^2} + \frac{\gamma-1}{2}\right)\sigma\, \delta_1' - m' \sin \mu'\, \delta_2' + km' \sin \mu'$$

$$+ \left(\frac{\cos \mu'}{v'} \phi' - \frac{1}{2\gamma c_v} S'_{,s}\right)\sigma .$$

Now [4;(5.7)]

$$\delta_i(s,T) = a_{ij}\, \zeta_{j,T} , \tag{2.32}$$

where

$$||a_{ij}(s,T)|| = \lambda \left\| \begin{matrix} \sin \mu & -m^2 \cos \mu \\ (\frac{1}{m^2} - 1)\cos \mu & \sin \mu \end{matrix} \right\| . \tag{2.33}$$

Thus

$$\sigma_{,s}(s,0) = \left(\frac{3-\gamma}{2} - \frac{2}{m'^2}\right) \sin \mu'\, \zeta'_{1,T} + \left(\cos 2\mu' + \frac{\gamma-1}{2}\sigma^2\right)m'\, \zeta'_{2,T}$$

$$+ km' \sin \mu' + \sigma\left(\frac{\cos \mu'}{v'} \phi - \frac{1}{2\gamma c_v} S'_{,s}\right). \tag{2.34}$$

From (2.24), (2.25), (2.22), (2.23) and (2.34), we have

$$\delta_i(s,0) = d_{ij}\, \zeta'_{j,T} + e_i k + f_i\left(\frac{\cos \mu'}{v'} \phi' - \frac{1}{2\gamma c_v} S'_{,s}\right), \tag{2.35}$$

where

$$||d_{ij}|| = \left\| \begin{matrix} \sin\mu' \dfrac{6\sigma^2 + 8\cos^2\mu' + 1-\gamma}{2\gamma\sigma^2 + 1-\gamma} & \dfrac{4\cos^2\mu' - 2\sigma^2 - 1+\gamma}{2\gamma\sigma^2 + 1-\gamma}\, m'\sigma \\ \\ \dfrac{\sin 2\mu}{2\sin\mu'}\left\{\dfrac{1}{m'^2} - 1 + \dfrac{2\sin^2\mu'}{[2+(\gamma-1)\sigma^2]m'^2}[(3-\gamma)m'^2-4]\right\}, & \dfrac{\sin^2\mu[4\cos 2\mu' + 3(\gamma-1)\sigma^2 + 2]}{2\cos\mu'\,[2+(\gamma-1)\sigma^2]} \end{matrix} \right\| \tag{2.36}$$

and

$$\left\|\begin{matrix} e_1 & f_1 \\ e_2 & f_2 \end{matrix}\right\| = \left\| \begin{matrix} \dfrac{4\gamma m'\sin\mu'}{2\gamma\sigma^2+1-\gamma} \quad , & \dfrac{4\sigma^2}{2\gamma\sigma^2+1-\gamma} \\ \\ 1 - \eta\, \dfrac{\cos^2\mu}{\cos^2\mu'} + \dfrac{2\sin^2\mu}{2+(\gamma-1)\sigma^2} \cdot \dfrac{m'\sin\mu'}{\sigma} \quad , & \dfrac{2\sin 2\mu}{2+(\gamma-1)\sigma^2} \end{matrix} \right\| \tag{2.37}$$

$$= \left\| \begin{matrix} \dfrac{2}{\gamma+1} m^2 \sin 2\mu \quad , & \dfrac{4\sigma^2}{2\gamma\sigma^2+1-\gamma} \\ \\ \dfrac{2}{\gamma+1}\left(1 - \dfrac{1}{m^2} + \dfrac{\sin^2\mu}{\sigma^2}\right) \quad , & \dfrac{2\sin 2\mu}{2+(\gamma-1)\sigma^2} \end{matrix} \right\| .$$

Differentiating (2.35), we have

$$\delta_{i,s^n}(s,0) = d_{ij}\,\zeta'_{j,s^n\mathrm{T}} + e_i\, k_{,s^n} + f_i\left(\frac{\cos\mu'}{v'}\,\phi_{,s^n} - \frac{1}{2\gamma c_v} S'_{,s^{n+1}}\right) + g_i^{(n)} \tag{2.38}$$

where

$$g_i^{(n)}(s,0) = g_{i,s}^{(n-1)} + d_{ij,s}\,\zeta'_{j,s^{n-1}\mathrm{T}} + e_{i,s} k_{,s^{n-1}} + \left(\frac{f_i\cos\mu'}{v'}\right)_{,s} \phi'_{,s^{n-1}} - \frac{f_{i,s}}{2\gamma c_v} S'_{,s^n} \tag{2.39}$$

and

$$g_i^{(0)}(s,0) = 0 . \tag{2.40}$$

Now [4; (7.5) and (7.6)]

$$\zeta_{i,T^n}(s,T) = A^{(n)}_{ij}\,\delta_{j,s^{n-1}} + P^{(n)}_i \tag{2.41}$$

where

$$P^{(n)}_i(s,T) = P^{(n-1)}_{i,T} + A^{(n-1)}_{ij,T}\,\delta_{j,s^{n-2}} + A^{(n-1)}_{ij}\sum_{r=1}^{n-1}\binom{n-1}{r} A_{jk,s^r}\,\delta_{k,s^{n-1-r}}$$

$$-\epsilon A^{(n-1)}_{ij}\,C_{j,Ts^{n-2}} \tag{2.42}$$

and

$$P^{(1)}_i(s,T) = 0\ . \tag{2.43}$$

Since

$$\zeta_{j,s^nT}(s,0) = \delta_{j,s^{n-1}T} + \epsilon C_{j,s^{n-1}T} = a^{(n)}_{ji}(\zeta_{i,T^{n+1}} - P^{(n)}_i) + \epsilon C_{j,s^{n-1}T}, \tag{2.44}$$

$$\delta_{i,s^n}(s,0) = d_{ij}\,a'^{(n)}_{j\ell}\,\zeta'_{\ell,T^{n+1}} + e_i k_{,s^n} + f_i\left(\frac{\cos\mu'}{v'}\phi'_{,s^n} - \frac{1}{2\gamma c_v}S'_{,s^{n+1}}\right) + h^{(n)}_i, \tag{2.45}$$

where

$$h^{(n)}_i(s,0) = g^{(n)}_i - d_{ij}(a^{(n)}_{j\ell}P'^{(n)}_\ell - \epsilon C'_{j,s^{n-1}T})\ . \tag{2.46}$$

From (2.45) and (2.41), we have the main equations

$$\zeta_{i,T^n}(s,0) = A^{(n)}_{ij}d_{jr}a'^{(n-1)}_{r\ell}\,\zeta'_{\ell,T^n} + A^{(n)}_{ij}e_j k_{,s^{n-1}} + A^{(n)}_{ij}f_j\left(\frac{\cos\mu'}{v'}\phi'_{,s^{n-1}} - \frac{1}{2\gamma c_v}S'_{,s^n}\right)$$

$$+ q^{(n)}_i\ , \tag{2.47}$$

where

$$q^{(n)}_i(s,0) = A^{(n)}_{ij}\,h^{(n-1)}_i + P^{(n)}_i\ . \tag{2.48}$$

A polynomial $r(s,0)$ of ζ_{j,T^ℓ}, $k_{,s^{\ell-1}}$, $\phi_{,s^{\ell-1}}$ and $S_{,s^\ell}$ for $\ell = 1, 2, \ldots, n$ without constant term and whose coefficients are functions of flow variables will be denoted by

$$r(s,0) = r(\zeta_{j,T^\ell}, k_{,s^{\ell-1}}, \phi_{,s^{\ell-1}}, S_{,s^\ell}, 1 \le \ell \le n)\ . \tag{2.49}$$

Then

$$g_i^{(1)}(s,0) = g_i^{(1)}(\zeta'_{j,s},\ k,\ \phi',\ S'_{,s}) \ , \tag{2.50}$$

since

$$\mu_{,s}(s,0) = \mu(\zeta'_{j,s},\ k,\ \phi', S'_{,s}), \quad \mu'_{,s}(s,0) = \mu'(\zeta'_{j,s},\ k,\ \phi', S'_{,s}) \ ,$$

$$m'_{,s}(s,0) = m'_{,s}(\zeta'_{j,s},\ k,\ \phi',\ S'_{,s}), \quad \zeta'_{j,T}(s,0) = \zeta'_{j,T}(\zeta'_{j,s}) \ ,$$

and by (2.28)

$$v_{,s}(s,0) = v_{,s}(\zeta'_{j,s},\ k,\ \phi',\ S'_{,s}) \ .$$

Since [4; (5.12) to (5.15)]

$$c_{,T}(s,T) = \frac{r-1}{2}\, c\, \zeta_{1,T} \tag{2.51}$$

$$m_{,T}(s,T) = -(\frac{1}{m} + \frac{r-1}{2}\, m)\zeta_{1,T} - \frac{\epsilon}{c} \ , \tag{2.52}$$

$$\lambda_{,T}(s,T) = \cos\mu\, \zeta_{2,s} \ , \tag{2.53}$$

and

$$\alpha_{,T}(s,T) = \frac{\sin\mu}{\lambda}\zeta_{2,s} \ ,$$

we have

$$h_i^{(1)}\ (s,0) = h_i^{(1)}\ (\zeta'_{j,s},\ k,\ \phi',\ S'_{,s}) \tag{2.55}$$

and

$$P_i^{(2)}(s,0) = P_i^{(2)}(\zeta'_{j,s},\ k,\ \phi',\ S'_{s}) \ . \tag{2.56}$$

Therefore

$$q_i^{(2)}(s,0) = q_i^{(2)}(\zeta'_{j,s},\ k,\ \phi',\ S'_{,s}) \ . \tag{2.57}$$

An induction argument shows that

$$q_i^{(n)}(s,0) = q_i^{(n)}\ (\zeta'_{j,s^{\ell-1}},\ k_{,s^{\ell-2}},\ \phi'_{,s^{\ell-2}},\ S'_{,s^{\ell-1}},\ 2 \leq \ell \leq n) \ . \tag{2.58}$$

Equations (2.47) extend the previous results obtained by Thomas [6], Brown [1] and Taub [4] to the general stationary or pseudo-stationary flows. The coefficients

$A_{ij}^{(n)} e_j$ have been denoted as $A_i^{(n)}$ in [4; (10.9) and (10.10)] and $A_{2j}^{(n)} e_j$ as $G_{n-1}(m',\alpha)$ in [6; (19)], provided that the flow ahead of the shock is uniform, i.e., $u_1(s,T) =$ constant and $u_2(s,T) =$ constant for $T < 0$. In [7], Thomas showed that for $m(s,0) \geq 1$

$$A_{2j}^{(n)} e_j(s,0) = G_{n-1}(m',\alpha) \neq 0 \qquad \text{for } n = 1,2, \ldots, n, \tag{2.59}$$

but if $\alpha = \alpha_1$ is such that $m(s,0) < 1$, then any neighborhood of α_1 contains a value of α at which

$$G_{n-1}(m',\alpha) = 0 \tag{2.60}$$

for some positive integer n.

(D) Simple consequences of (2.47). Let the flow ahead of the shock be uniform. Then

$$\zeta'_{i,T}(s,T) = 0 \tag{2.61}$$

and (2.47) for $i = 2$ becomes

$$\zeta_{2,T^n}(s,0) = A_{2j}^{(n)} e_j \, k_{,s^{n-1}} + q_2^{(n)} . \tag{2.62}$$

If $m(0,0) \geq 1$ and $K(0,T) = 0$ for $T > 0$, then by (2.59), (2.62) and (2.58)

$$k(0,0) = q_2^{(2)} = k_{,s} = q_2^{(3)} = k_{,s^2} = \ldots = k_{,s^n} = \ldots = 0 . \tag{2.63}$$

It is easy to see that (2.63) still holds if we assume only that the continuity and differentiability requirements on the functions involved in obtaining (2.47) for $s > 0$ are satisfied and

$$\lim_{s \to 0} m(s,0) \geq 1, \quad \lim K_{,T^n}(s,0) = \lim K'_{,T^n}(s,0) = 0 ,$$

$$\lim \zeta'_{1,T^{n+1}}(s,0) = \lim \phi'_{,s^n}(s,0) = \lim S'_{,s^{n+1}}(s,0) = 0 \text{ and}$$

$$\lim k_{,s^n}(s,0) = k_{,s^n}(0,0) \qquad \text{for } n = 0, 1, \ldots, n .$$

If the flow ahead of the shock is uniform and after the shock supersonic near the vertex, then (2.63) gives proofs to the following results at the vertex. (a) The attached shock to a cone at its vertex is itself a cone. (b) The attached shock to a wedge is straight. (c) In regular reflection, the reflected shock behind a straight incident shock is straight.

From (2.17),

$$\rho_{,s}(s,0) = \frac{4\rho\sigma_{,s}}{(2+r\sigma^2-\sigma^2)\sigma} = \frac{4m'\rho\sin\mu'}{(2+r\sigma^2-\sigma^2)\sigma}\,k\ . \tag{2.64}$$

By (2.30) and (2.45),

$$S_{,s}(s,0) = c_v(\gamma\zeta_{1,s} - \frac{\gamma\rho_{,s}}{\rho}) = c_v\gamma k(e_1 - \frac{4m'\sin\mu'}{(2+\gamma\sigma^2-\sigma^2)\sigma})\ . \tag{2.65}$$

If, moreover, $\phi(s,0) = 0$, then by (2.45)

$$0 = \delta_i'(s,0) = d_{ij}'\ a_{j\ell}\zeta_{\ell,T} + e_i'k - \frac{f_i'}{2\gamma c_v}S_{,s}\ . \tag{2.66}$$

Therefore

$$0 = (d_{ij}'e_j+e_i' - \frac{f_i}{2}e_1 + \frac{2f_i'm'\sin\mu'}{(2+\gamma\sigma^2-\sigma^2)\sigma})k(s,0) = X_i k\ , \text{ say.} \tag{2.67}$$

X_i are functions of $m'(s,0)$ and $\sin\mu'(s,0)$, whose coefficients are constants. Since the Jacobian of X_1 and X_2, with respect to m' and $\sin\mu'$, is not zero as $\mu' = 0$, X_1 and X_2 do not have a relation between them. For each interval I, if $X_1(s,0) \neq 0$ or $X_2(s,0) \neq 0$ for some $s_0 \in I$, then $k(s_0,0) = 0$ by (2.67), and if $X_1(s,0) = X_2 = 0$ for all $s \in I$, then $m'(s,0) =$ constant and $\mu'(s,0) =$ constant for $s \in I$ and the flow is stationary and $k(s,0) = 0$ for $s \in I$. Therefore, for each interval I, there always exists an $s_0 \in I$ such that $k(s_0,0) = 0$. By the continuity of k, we prove that

$$k(s,0) = 0 \qquad \text{for all } s\ . \tag{2.68}$$

Thus, if the flow ahead of the shock is uniform and the flow after shock is irrotational, then the shock is straight and the flow after the shock is uniform also. This result can also be obtained from a much stronger result of Taub [University of Illinois lecture notes]. Every irrotational non-isentropic pseudo-stationary flow has constant particle velocity. But the proof given here is simpler.

3. Normal Shocks

In this section we are going to discuss a normal shock attached to a solid boundary with continuous tangent. The curve s = 0 will be the boundary which has continuous curvature everywhere except at (0,0) and there it may have

a finite jump. The shock $a_i(s,0)$, $s \geq 0$, is regular everywhere except, possibly, at $(0,0)$, where $\lim\limits_{s\to 0} k(s,0) = k(0,0)$ may be infinite. The flow in front of the shock is regular off the boundary. We assume that

$$\lim_{s\to 0} f(s,0) = f(0,0) \quad \text{and} \quad \lim f'(s,0) = f'(0,0) \tag{3.1}$$

for all quantities f used below, and that

$$\phi'(0,0),\ S'_{,s}(0,0),\ K'(0,0) \text{ and } p'_{,T}(0,0) \text{ are finite.} \tag{3.2}$$

Since the shock is normal and the boundary is continuous at $(0,0)$,

$$\alpha(0,0) = \frac{\pi}{2},\ \sin\mu(0,0) = \sin\mu'(0,0) = 0. \tag{3.3}$$

(A) Limiting case of (2.47) for $n = 1$. When $n = 1$ and $i = 2$, (2.47) becomes

$$\zeta_{2,T}(s,0) = A_{2j}\, d_{j\ell}\, \zeta'_{\ell,T} + A_{2j}\, e_j k + A_{2j} f_j \left(\frac{\cos\mu'}{v'}\,\phi' - \frac{1}{2\gamma c_v}\, S'_{,s}\right). \tag{3.4}$$

Now

$$\lim_{s\to 0} A_{2j} d_{j1}(s,0) = 0,$$

$$\lim A_{2j}\, d_{j2}(s,0) = \frac{m'^2(2m'^2 - 3 - r)}{m^2(2\gamma m'^2 + 1 - \gamma)}\,(0,0),$$

$$\lim_{s\to 0} A_{2j}\, e_j k(s,0) = -\frac{2(2m^2+1)}{(\gamma+1)m^2} \lim k \sin\mu = -\frac{2(2m^2+1)v'}{(\gamma+1)m^2 v} \lim(\zeta'_{2,s} - \mu'_{,s})\sin\mu'$$

$$= -\frac{2v'(2m^2+1)}{(\gamma+1)m^2 v} \lim(\cos\mu')_{,s} = \frac{v'^2(2m^2+1)}{(\gamma+1)m^2 v U'_n} \lim(\cos^2\mu')_{,s}$$

$$\lim A_{2j} f_j(s,0) = -\frac{4m'^2 v'}{(2\gamma m'^2 + 1 - \gamma)m^2 U'_n}(0,0).$$

From (3.4), (2.8), (2.9) and (2.10) we have

$$K(0,0) = \frac{m'^2}{m^2(2\gamma m'^2+1-\gamma)}\left[(2m'^2-3-\gamma)K' - \frac{2}{\gamma c_v}S'_{,s} - \frac{4\phi'U'_n}{v'^2}\right](0,0)$$
$$- \frac{v'(2m^2+1)}{(\gamma+1)m^2v}(0,0)\lim(\cos^2\mu')_{,s}(s,0) . \tag{3.5}$$

$$\left(\frac{2}{m'^2} + \gamma - 1\right)K(0,0) \geq \left[(2m'^2 - 3 - \gamma)K' - \frac{2}{\gamma c_v}S'_{,s} - \frac{4\phi'U'_n}{v'^2}\right](0,0) . \tag{3.6}$$

(3.5) and (3.6) hold for a general stationary or pseudo-stationary flow. The equality of (3.6) holds if and only if

$$\lim_{s\to 0} k \sin\mu\,(s,0) = 0 . \tag{3.7}$$

(B) Brown's result. Let the flow be stationary and iso-energetic. Then by (2.29) and (2.31)

$$\frac{1}{\gamma c_v}S'_{,s} = -\frac{\gamma-1}{c^2}U'_n\phi' , \tag{3.8}$$

and (3.6) reduces to

$$\left(\frac{2}{m'^2} + \gamma - 1\right)K(0,0) \geq (2m'^2 - 3 - \gamma)K'(0,0) - 2\left(\frac{v-1}{c^2} + \frac{2}{v'^2}\right)\phi'U'_n(0,0) \tag{3.9}$$

Put

$$\Gamma = \frac{-\phi'}{v'\xi'_{2,T}}(0,0) .$$

Then (3.9) becomes

$$\left(\frac{2}{m'^2} + \gamma - 1\right)K(0,0) \geq (2m'^2 - 3 - \gamma)K' + 2[(\gamma-1)m'^2 + 2]\Gamma K' . \tag{3.10}$$

Equation (3.10) is the same as Brown's [1;(26)] , which should be

$$K(0,0) \geq \frac{2M^2}{(\gamma-1)M^2+2}\left(M^2 - \frac{\gamma+3}{2}\right)K'(0,0) + 2(\gamma-1)\,M^2\,K'\;(0,0) . \tag{3.11}$$

(C) Suppose that the flow is irrotational isentropic ahead of the shock. Then (3.6) reduces to

$$\left(\frac{2}{m'^2} + \gamma - 1\right)K(0,0) \geq (2m'^2 - 3 - \gamma)K'(0,0), \tag{3.12}$$

which extends Lin and Rubinov's result [3;(4.7)] to pseudo-stationary flows. If, moreover, k(0,0) is finite, then (3.7) is satisfied and

$$\left(\frac{2}{m'^2} + \gamma - 1\right)K(0,0) = (2m'^2 - 3 - \gamma)K'(0,0), \tag{3.13}$$

which was proved by Tsien [9;(16)] and Lin and Rubinov [3;(4.6)] for stationary flows. Let $M_c^2 = \frac{1}{2}[v+1 + \sqrt{(v+1)^2+4}]$. Then (3.12) gives $m'(0,0) \leq M_c$ when $K(0,0) = K'(0,0) \geq 0$ and $m'(0,0) \geq M_c$ when $K(0,0) = K'(0,0) \leq 0$.

(D) Suppose that $K(0,0) < \infty$. Then, from (3.5), we have

$$\lim\,(\cos^2\mu')_{,s}(s,0) = -2\beta \tag{3.14}$$

is finite and $\beta \geq 0$. Hence,

$$\cos^2\mu'(s,0) = -1 - 2\beta s + o(s)\ , \tag{3.15}$$

where o(f) means that

$$\lim_{s\to 0} \frac{o(f)}{f} = 0\ .$$

Hence,

$$\sin^2\mu'(s,0) = 2\beta s + o(s)\ .$$

Since

$$\lim\, k \sin\mu'(s,0) = \lim\,(\int_{2,s} - \mu'_{,s}) \sin\mu'(s,0) = \lim\,(\cos\mu)_{,s}(s,0),$$

$$k(s,0) = \pm\sqrt{\frac{\beta}{2s}} + o\left(\frac{1}{\sqrt{s}}\right)\ ,$$

and then

$$\alpha(s,0) = \frac{\pi}{2} \pm \sqrt{2\beta s} + o(\sqrt{s})\ .$$

$$\lambda_2(s,0) = \sin\alpha = \sin(\tfrac{\pi}{2} \pm \sqrt{2\beta s} + o(\sqrt{s})) = \cos(\sqrt{2\beta s} + o(\sqrt{s})) = 1 + O(s),$$

where $O(f)$ means that $\frac{O(f)}{f}$ is bounded as $s \to o$.

$$\lambda_1(s,0) = \cos\alpha = \sin(\mp\sqrt{2\beta s} + o(\sqrt{s})) = \mp\sqrt{2\beta s} + o(\sqrt{s}) .$$

Hence,

$$a_2(s,0) = s + O(s^2) . \tag{3.16}$$

$$a_1(s,0) = \mp \tfrac{2}{3}\sqrt{2\beta}\, s^{3/2} + o(s^{3/2}) = \mp\sqrt{\tfrac{8\beta}{9}}\, s^{3/2} + o(s^{3/2}) . \tag{3.17}$$

Therefore the shock is approximately of the form

$$a_2 = (\tfrac{9}{8\beta}\, a_1{}^2)^{1/3} . \tag{3.18}$$

Let $\beta_1 = \sqrt{\frac{8\beta}{9}}$. Then $a_1 = \pm\beta_1\, a_2{}^{3/2}$ and from (3.5)

$$\left(\frac{2}{m'^2} + \gamma - 1\right)K(0,0) = (2m'^2 - 3 - \gamma)K' - \frac{2}{\gamma c_v} S'_{,s} - \frac{4\phi' U'_n}{v'^2} + \frac{9\beta_1^2}{4}\left(\frac{2\gamma m'^2 + 1 - \gamma}{2 + (\gamma-1)m'^2} + 2\right\}\frac{v'}{U'_n} \tag{3.19}$$

(3.17) and (3.18) extend Lin and Rubinov's result [3;(5.5) and (5.6)] to a general stationary and pseudo-stationary flow, and their equation (5.6) should be

$$\left\{2M_1^2 - (\gamma + 3)\right\}\frac{1}{r_1} - \left\{\frac{2}{M_1{}^2} + (\gamma - 1)\right\}\frac{1}{r_2} = \frac{9k^2}{2}\left\{1 + \frac{2\gamma M_1^2 - (\gamma-1)}{2[1 + (\gamma-1)M_1^2]}\right\}. \tag{3.20}$$

Where $M_1 = m'$, since in their case $\frac{v'}{U'_n}(0,0) = 1$.

4. Normal Shock Attached to a Straight Wall

In this section we will assume that the flow ahead of the shock is uniform and the continuity and differentiability requirements on functions involved below are assumed satisfied.

Then
$$\zeta_1'(s,T) = \zeta_{2,T}'(s,T) = 0 \tag{4.1}$$

$$\sigma_{,s}(s,0) = km' \sin \mu' \tag{4.2}$$

$$\zeta_{i,T}(s,0) = A_{ij}\, e_j\, k\ , \tag{4.3}$$

and
$$\delta_i(s,0) = e_i\, k\ , \tag{4.4}$$

where
$$e_1(s,0) = \frac{2}{\gamma+1} m^2 \sin^2 \mu\ , \tag{4.5}$$

and
$$e_2(s,0) = \frac{2}{\gamma+1}\left(1 - \frac{1}{m^2} + \frac{\sin^2\mu}{\sigma^2}\right)\ . \tag{4.6}$$

Since the shock is normal to a straight wall, by taking s = 0 to be the wall, we have

$$\alpha(0,0) = \frac{\pi}{2}, \qquad \zeta_2'(0,T) = \zeta_2(0,T) = 0,\ \text{or}\ \pi. \tag{4.7}$$

For simplicity, let $\zeta'_2(0,T) = \zeta'_2(0,T) = \pi$. Then

$$\mu(0,0) = \pi\ .$$

Suppose that $(\mu-\pi)_{,T^i}(0,0) = 0$, for $i = 0, 1, \ldots, h-1$. Then

$$(\sin \mu)_{,T^i}(0,0) = (\cos \mu \cdot \mu_{,T})_{,T^{i-1}} = 0 \tag{4.8}$$

and

$$\mu_{,T^h}(0,0) = -\alpha_{,T^h} = -\left(\frac{\sin \mu}{\lambda}\, \zeta_{2,s}\right)_{,T^{h-1}} = 0$$

by (2.54). Therefore, for n = 0, 1, 2, 3, ...,

$$(\mu-\pi)_{,T^n}(0,0) = A_{11,T^n} = A_{22,T^n} = 0\ . \tag{4.9}$$

From (4.4),

$$\delta_1(0,0) = 0\ .$$

Let

$$\delta_{1,T^i}(0,0) = 0, \qquad \text{for } i = 0, 1, 2, \ldots, h-1\ .$$

Then

$$0 = \xi_{2,T^{h+1}}(0,0) = (A_{2j}\ \delta_j)_{,T^h} = \sum_{i=0}^{h} \binom{h}{i} A_{2j,T^i}\ \delta_{j,T^{h-i}}$$

$$= \sum^{h}(i)A_{21,T^i}\ \delta_{1,T^{h-i}} = A_{21}\ \delta_{1,T^h} \ .$$

Since

$$A_{21}(0,0) \neq 0 \ , \qquad \delta_{1,T^h}(0,0) = 0 \ .$$

Hence

$$\delta_{1,T^n}(0,0) = 0 \quad \text{for } n = 0, 1, 2, \ldots \tag{4.10}$$

In the following, we will show, by induction, that

$$\delta_{1,s^{2i}T^n}(0,0) = \delta_{2,s^{2i+1}T^n} = 0 \tag{4.11}$$

for $i,n = 0, 1, 2, \ldots$. Suppose that for $i = 0, 1, \ldots, \ell$, $n = 0, 1, 2, \ldots$,

$$A_{11,s^{2i}T^n}(0,0) = 0 \ , \tag{4.12}$$

$$\delta_{1,s^{2i}T^n}(0,0) = 0 \ , \tag{4.13}$$

$$\delta_{2,s^{2-1}T^n}(0,0) = 0 \ , \tag{4.14}$$

$$m_{,s^{2i-1}T^n}(0,0) = 0 \ , \tag{4.15}$$

$$c_{,s^{2i-1}T^n}(0,0) = 0 \ , \tag{4.16}$$

$$(\mu - \pi)_{,s^{2i}T^n}(0,0) = 0 \ , \tag{4.17}$$

$$(\sin \mu)_{,s^{2i}T^n}(0,0) = 0 \ , \tag{4.18}$$

and

$$k_{,s^{2i-1}}(0,0) = 0 \ , \tag{4.19}$$

where $f_{,s^iT^j}(0,0) = 0$ if i or j is negative. To complete the induction, we need the following lemma.

Lemma. If $f_{,s^{2i+1}T^j}(0,0) = g_{,s^{2i+1}T^j} = 0$ for $i = 0, 1, \ldots, h$ and $j = 0, 1, \ldots, \ell$, then

$$(f^{-1})_{,s^{2i+1}T^j}(0,0) = (fg)_{,s^{2i+1}T^j} = 0 \quad .$$

The proof follows from

$$(f^{-1})_{,s^{2i+1}T^j}(0,0) = -\left(\frac{1}{f^2} f_{,s}\right)_{,s^{2i}T^j} = -\sum_{q=0}^{2i} \binom{2i}{q} \left[\left(\frac{1}{f^2}\right)_{,s^q} f_{,s^{2i+1-q}}\right]_{,T^j} = 0,$$

and

$$(fg)_{,s^{2i+1}T^j}(0,0) = (f_{,s}\, g + f g_{,s})_{,s^{2i}T^j} = 0 \; .$$

By the lemma,

$$v_{,s^{2i-1}T^n}(0,0) = (mc)_{,s^{2i-1}T^n} = 0 \qquad (4.20)$$

$$\sigma_{,s^{2i+1}}(0,0) = \frac{1}{c'}(kv \sin \mu)_{,s^{2i}} = \frac{1}{c'} \sum_{j=0}^{2i} \binom{2i}{j} (kv)_{,s^j} (\sin \mu)_{s^{2i-j}} = 0 \qquad (4.21)$$

$$(\cos \mu)_{,s^{2i+1}T^n}(0,0) = -(\sin \mu\, \mu_{,s})_{,s^{2i}T^n} = 0 \quad . \qquad (4.22)$$

Now

$$m^2 = \left(\frac{\gamma-1}{2\gamma} + \frac{(1+\gamma)^2}{2\gamma(2\gamma\sigma^2+1-\gamma)}\right) \frac{1}{\cos^2 \mu} \qquad (4.23)$$

we have

$$m^2_{,s^{2i+1}}(0,0) = 0 \,,$$

and

$$(2mm_{,s})_{,s^{2i}}(0,0) = 2m_{,s^{2i+1}} = 0 \quad . \qquad (4.24)$$

$$y_{,s^{2i+1}}(0,0) = \left[1 + \frac{2\gamma}{\gamma+1}(\sigma^2-1)\right]_{,s^{2i+1}} = 0 \; .$$

$$\eta_{,s^{2i+1}}(0,0) = (\gamma+1)\left[\frac{\sigma^2}{2+(\gamma-1)\sigma^2}\right]_{,s^{2i+1}} = 0 \; .$$

$$c^2_{,s^{2i+1}}(0,0) = \frac{\gamma p_1}{\rho_1}\left(\frac{y}{\eta}\right)_{,s^{2i+1}} = 0 \; .$$

Therefore,

$$c_{,s^{2i+1}}(0,0) = 0. \tag{4.25}$$

Let

$$M_{,s^{2\ell+1}T^j}(0,0) = c_{,s^{2\ell+1}T^j} = 0, \quad \text{for } j = 0,\ 1,\ \dots,\ h-1.$$

Then by (2.51) and (2.52)

$$\begin{aligned} m_{,s^{2\ell+1}T^h}(0,0) &= -[(\frac{1}{m} + \frac{\gamma-1}{2}m)\zeta_{1,T} + \frac{\epsilon}{c}]_{,s^{2\ell+1}T^{h-1}} \\ &= -\sum\binom{2\ell+1}{q}[(\frac{1}{m} + \frac{\gamma-1}{2}m)_{,s^q}\zeta_{1,Ts^{2\ell+1-q}}]_{,T^{h-1}} \\ &= -\sum\binom{2\ell+1}{q}[(\frac{1}{m} + \frac{\gamma-1}{2}m)_{,s^q}\delta_{1,Ts^{2\ell-q}}]_{,T^{h-1}} = 0\ , \end{aligned}$$

and

$$\begin{aligned} c_{,s^{2\ +1}T^h}(0,0) &= \frac{\gamma-1}{2}(c\,\zeta_{1,T})_{,s^{2\ell+1}T^{h-1}} = \frac{\gamma-1}{2}\sum\binom{2\ell+1}{q}[c_{,s^q}\zeta_{1,Ts^{2\ell+1-q}}]_{,T^{h-1}} \\ &= \frac{\gamma-1}{2}\sum\binom{2\ell+1}{q}[c_{,s^q}\delta_{1,Ts^{2\ell-q}}]_{,T^{h-1}} = 0\ . \end{aligned}$$

By (4.24) and (4.25), for $n = 0,\ 1,\ 2,\ \dots,$

$$m_{,s^{2\ell+1}T^n}(0,0) = c_{,s^{2\ell+1}T^n} = v_{,s^{2\ell+1}T^n} = 0\ , \tag{4.26}$$

and

$$(\frac{1}{1-m^2\cos^2\mu})_{,s^{2i+1}T^n}(0,0) = 0\ , \tag{4.27}$$

since

$$(1-m^2\cos^2\mu)_{,s^{2i+1}T^n}(0,0) = 0\ .$$

$$\begin{aligned} A_{12,s^{2i+1}}(0,0) = (\frac{m^2\cos\mu}{1-m^2\cos^2\mu})_{,s^{2i+1}} &= \sum\binom{2i+1}{q}(\frac{m^2}{1-m^2\cos^2\mu})_{,s^q}(\cos\mu)_{,s^{2i+1-q}} \\ &= 0\ . \end{aligned}$$

$$0 = \delta_{1,s^{2\ell}T}(0,0) = (a_{1j}\delta_j)_{,s^{2\ell+1}} = \sum\binom{2\ell+1}{q}A_{1j,s^q}\delta_{j,s^{2\ell+1-q}} = A_{12}\delta_{2,s^{2\ell+1}}\ \cdot$$

Hence,

$$\delta_{2,s^{2\ell+1}}(0,0) = 0. \tag{4.28}$$

Let

$$\delta_{2,s^{2\ell+1}T^j}\ (0,0) = \lambda_{,s^{2\ell+1}T^j} = 0, \quad \text{for } j = 0,\ 1,\ \ldots,\ h-1 .$$

Then

$$\begin{aligned} c_{2,s^{2i+1}T^j}\ (0,0) &= -\left(\frac{\lambda \cos \mu}{v}\right)_{,s^{2i+1}T^j} = -\Sigma\binom{2i+1}{q}\left[\lambda_{,s^q}\left(\frac{\cos \mu}{v}\right)_{,s^{2i+1-q}}\right]_{,T^j} \\ &= 0 . \end{aligned}$$

Since [4;(15.14)]

$$\lambda_{,T}(s,T) = \cos \mu\, \zeta_{2,s} . \tag{4.29}$$

Hence,

$$\begin{aligned} \lambda_{,s^{2\ell+1}T^h}\ (0,0) &= (\cos \mu\, \zeta_{2,s})_{,s^{2\ell+1}T^{h-1}} = [\cos \mu(\delta_2+c_2)]_{,s^{2\ell+1}T^{h-1}} \\ &= \Sigma\binom{2\ell+1}{q}[(\cos \mu)_{,s^q}(\delta_2+c_2)_{,s^{2\ell+1-q}}]_{,T^{h-1}} = 0 , \end{aligned}$$

and

$$c_{2,s^{2\ell+1}T^h}(0,0) = \Sigma\binom{2\ell+1}{q}\left[\lambda_{,s^q}\left(\frac{\cos \mu}{v}\right)_{,s^{2i+1-q}}\right]_{,T^h} = 0 .$$

Now

$$A_{12,s^{2i+1}T^h}\ (0,0) = \left[\frac{m^2 \cos \mu}{\lambda(1-m^2\cos^2\mu)}\right]_{,s^{2i+1}T^h} = 0 ,$$

and

$$\begin{aligned} 0 = \delta_{1,s^{2\ell}T^{h+1}}\ (0,0) &= (\zeta_{1,T})_{,s^{2\ell+1}T^h} = (A_{1j}\delta_j)_{,s^{2\ell+1}T^h} \\ &= \Sigma\binom{2\ell+1}{q}(A_{1j,s^q}\, \delta_{j,s^{2\ell+1-q}})_{,T^h} = [A_{12}\, \delta_{2,s^{2\ell+1}}]_{,T^h} = A_{12}\, \delta_{2,s^{2\ell+1}T^h} . \end{aligned}$$

Therefore, for $i = 0,\ 1,\ 2,\ \ldots, \ell$ and $n = 0,\ 1,\ 2,\ \ldots,$

$$\delta_{2,s^{2\ell+1}T^n}\ (0,0) = \lambda_{,s^{2\ell+1}T^n} = A_{12,s^{2i+1}T^n} = c_{2,s^{2i+1}T^n} = 0 . \tag{4.30}$$

Since

$$(\sin^2\mu)_{,s^{2i+1}}\ (0,0) = 2(\sin \mu \cos \mu\, \mu_{,s})_{,s^{2i}} = 0 ,$$

we have

$$\left(\frac{\sin^2\mu}{\sigma^2}\right)_{,s^{2i+1}}(0,0) = 0$$

by the lemma, and then

$$b_{2,s^{2i+1}}(0,0) = \frac{2}{\gamma+1}\left(1 - \frac{1}{m^2} + \frac{\sin^2\mu}{\sigma^2}\right)_{,s^{2i+1}} = 0 \,. \qquad (4.31)$$

Hence,

$$0 = (kb_2)_{,s^{2\ell+1}}(0,0) = \Sigma\binom{2\ell+1}{q} k_{,s^q}\, b_{2,s^{2\ell+1-q}} = k_{,s^{2\ell+1}} \,. \qquad (4.32)$$

Since

$$\mu_{,s^{2i+2}}(0,0) = (\delta_2 + C_2 - k)_{,s^{2i+1}} = 0 \,. \qquad (4.33)$$

$$(\sin\mu)_{,s^{2\ell+2}}(0,0) = \Sigma\binom{2\ell+1}{q}(\cos\mu)_{,s^q}\,\mu_{,s^{2\ell+2-q}} = \cos\mu\mu_{,s^{2\ell+2}} = 0.$$

$$A_{11,s^{2\ell+2}}(0,0) = \Sigma\binom{2\ell+1}{q}\left(\frac{1}{1-m^2\cos^2\mu}\right)_{,s^q}(\sin\mu)_{,s^{2\ell+2-q}} = 0 \,.$$

Hence,

$$0 = \delta_{2,s^{2\ell+1}T}(0,0) = (\zeta_{2,s} - C_2)_{,s^{2\ell+1}T} = \zeta_{2,s^{2\ell+2}T}$$

$$= (A_{2j}\delta_j)_{,s^{2\ell+2}} = \Sigma\binom{2\ell+2}{q}(A_{21,s^q}\,\delta_{1,s^{2\ell+2-q}} + A_{22,s^q}\,\delta_{2,s^{2\ell+2-q}})$$

$$= A_{21}\,\delta_{1,s^{2\ell+2}} \,.$$

Therefore

$$\delta_{1,s^{2\ell+2}}(0,0) = 0 \,. \qquad (4.34)$$

Let

$$\delta_{1,s^{2\ell+2}T^j}(0,0) = \mu_{,s^{2\ell+2}T^j} = 0, \text{ for } j = 0, 1, \ldots, h-1.$$

Then

$$(\sin\mu)_{,s^{2\ell+2}T^j}(0,0) = \Sigma\binom{2\ell+1}{q}[(\cos\mu)_{,s^q}\,\mu_{,s^{2\ell+2-q}}]_{,T}$$

$$= (\cos\mu\mu_{,s^{2\ell+2}})_{,T^j} = 0 \,,$$

$$A_{11,s^{2\ell+2}T^j}(0,0) = \left[\frac{\sin\mu}{\lambda(1-m^2\cos^2\mu)}\right]_{,s^{2\ell+2}T^j} = 0 \,,$$

and

$$\mu_{,s^{2\ell+2}T^h}(0,0) = (\zeta_{2,T} - \alpha_{,T})_{,s^{2\ell+2}T^{h-1}} = (A_{2j}\delta_j)_{,s^{2\ell+2}T^{h-1}}$$

$$-\left(\sin\mu\,\frac{\zeta_{2,s}}{\lambda}\right)_{,s^{2\ell+2}T^{h-1}} = \sum\binom{2\ell+2}{q}\left[(A_{2j,s^q}\,\delta_{j,s^{2\ell+2-q}})\right.$$

$$\left.-(\sin\mu)_{,s^q}\left(\frac{\zeta_{2,s}}{\lambda}\right)_{,s^{2\ell+2-q}}\right]_{,T^{h-1}} = 0\ .$$

Therefore

$$(\sin\mu)_{,s^{2\ell+2}T^h}(0,0) = A_{11,s^{2\ell+2}T^h} = 0\ .$$

Thus

$$0 = \delta_{2,s^{2\ell+1}T^{h+1}}(0,0) = (\zeta_{2,s} - C_2)_{,s^{2\ell+1}T^{h+1}} = (A_{2j}\delta_j)_{,s^{2\ell+2}\ h}$$

$$= \sum\binom{2\ell+2}{q}(A_{21,s^q}\,\delta_{1,s^{2\ +2-q}} + A_{22,s^q}\,\delta_{2,s^{2\ell+2-q}})_{,T^h}$$

$$= (A_{21}\,\delta_{1,s^{2\ell+2}})_{,T^h} = A_{21}\,\delta_{1,s^{2\ell+2}T^h}\ .$$

Hence,

$$\delta_{1,s^{2\ell+2}T^h}(0,0) = 0\ .$$

By (4.43) and (4.44), we have for $i = 0, 1, \ldots, \ell$, $n = 0, 1, 2, \ldots.$,

$$\mu_{,s^{2\ell+2}T^n}(0,0) = \delta_{1,s^{2\ell+2}T^n} = A_{11,s^{2\ell+2}T^n} = (\sin\mu)_{,s^{2\ell+2}T^n} = 0\ . \qquad (4.35)$$

By (4.35), (4.32), (4.30), (4.26), (4.10), (4.9) and (4.8) we complete the induction. Therefore, (4.11) and (4.19) are true.

Remark. Since $b_1(s,0) = \dfrac{4m^2\sin\mu\cos\mu}{\gamma+1}$, we have for $n = 0, 1, 2, \ldots$,

$$b_{1,s^{2n+1}}(0,0) = \frac{4}{\gamma+1}(m^2\sin\mu\cos\mu)_{,s^{2n+1}} = 0\ ,$$

by (4.22), (4.24) and (4.18). Relation (4.31) says

$$b_{2,s^{2n+1}}(0,0) = 0\ .$$

Therefore, we are unable to determine $k_{,s^{2n}}(0,0)$ from

$$\delta_{1,s^{2n}}(0,0) = (kb_1)_{,s^{2n}} = 0\ ,\ \text{and}\ \delta_{2,s^{2n+1}}(0,0) = (kb_2)_{,s^{2n+1}} = 0\ ,$$

since in both of these equations, the coefficients of $k_{,s^{2i}}$ are zero.

Approximation Forms

In this section we assume that the flow ahead of the shock is uniform and the hock $T = 0$ is normal to the solid boundary $s=0$. Thomas [9] proved that if the shock as finite curvature at the point of attachment $(0,0)$, then the flow is not stable. ere we assume that $k(0,0) = \infty$, $\zeta_{2,T}(0,0) < \infty$, and

$$\lim f_{,s^iT^j}(s,0) = f_{,s^iT^j}(0,0) \tag{5.1}$$

or $i,j = 0$ or 1, where $f_{,s^iT^j}(0,0)$ may be infinite. Since the shock is normal,

$$\sin \mu(0,0) = 0 \, , \qquad \alpha(0,0) = \frac{\pi}{2} \quad . \tag{5.2}$$

rom (4.3)

$$\zeta_{2,T}(s,0) = A_{ij}\, e_j\, k \quad . \tag{5.3}$$

$$k(s,0) = \frac{\gamma+1}{2} \frac{(1-m^2 \cos^2 \mu)\, K(s,0)}{(m^2-1)\cos \mu \sin^2\mu + \sin \mu(1 - \frac{1}{m^2} + \frac{\sin^2 \mu}{\sigma^2}} \quad . \tag{5.4}$$

ence,

$$\sin \mu\, k(s,0) \cong -\frac{(\gamma+1)m^2 K(s,0)}{2(2m^2+1)} \quad . \tag{5.5}$$

ince [4; p. 315]

$$\mu_{,s}(s,0) = -k\left[\frac{c_1 \sigma \cos \mu}{v} + \left(\gamma-1-\frac{2}{\sigma^2}\right) \frac{\sin^2\mu}{\gamma+1}\right] - \frac{\epsilon \cos \mu}{v} \, , \tag{5.6}$$

$$\sin \mu\, k(s,0) \cong \frac{m^2(\gamma+1)v'}{2(2m^2+1)v} K(s,0) \quad . \tag{5.7}$$

et

$$\zeta_{2,T}(s,0) = K(s,0) = K_0 s^{\bar{\alpha}} \, , \quad K_0 \neq 0, \; 0 \leq \bar{\alpha} \leq 1 \Rightarrow 0 \leq \bar{\alpha} \leq 1, \text{ and } \acute{\mu}(0,0)=\pi. \tag{5.8}$$

hen

$$\cos \mu(s,0) \cong -1 + \frac{m^2(\gamma+1)v'}{2(2m^2+1)(\bar{\alpha}+1)v} K_0 s^{\bar{\alpha}+1}$$

$$\sin \mu(s,0) \cong \pm \left\{\frac{m^2(\gamma+1)\; v'\, K_0}{(2m^2+1)(\bar{\alpha}+1)v}\right\}^{\frac{1}{2}} s^{\frac{\bar{\alpha}+1}{2}} \tag{5.9}$$

and

$$K_0 \geq 0 \quad . \tag{5.10}$$

Hence,

$$k(s,0) \cong \mp \left\{ \frac{m^2(\gamma+1)K_0 v(\bar{\alpha}+1)}{4(2m^2+1)} \right\}^{1/2} s^{\bar{\alpha}-1/2} \quad . \tag{5.11}$$

From (5.8),

$$\zeta_2(s,T) = \zeta_2(s,0) - K_0\, s^{\bar{\alpha}}\, T + \ldots \tag{5.12}$$

Now

$$\zeta_{2,s}(s,0) = C_2 + c_2 k \tag{5.13}$$

$$\approx \pm \frac{1-m^2}{m} \left\{ \frac{(\bar{\alpha}+1)K_0 v}{(\gamma+1)(2m^2+1)v'} \right\}^{1/2} s^{\bar{\alpha}-1/2} \quad .$$

Hence,

$$\zeta_2(s,0) = \pi + \frac{2(1-m^2)}{m} \left\{ \frac{K_0\ v}{(\bar{\alpha}+1)(\gamma+1)(2m^2+1)c_1 \sigma \nRightarrow v'} \right\}^{1/2} s^{\bar{\alpha}+1/2} \tag{5.14}$$

and

$$\zeta_2(s,T) = \frac{2(1-m^2)}{m} \left\{ \frac{K_0\ v}{(\bar{\alpha}+1)(\gamma+1)(2m^2+1)c_1 \sigma \nRightarrow v'} \right\}^{1/2} s^{\bar{\alpha}+1/2} - K_0 s^{\bar{\alpha}} T \ . \tag{5.15}$$

$$a_1(s,T) = a_1(s,0) + \int_0^T \cos \zeta_2 \, dT \approx a_1(s,0) - T \tag{5.16}$$

$$a_2(s,T) = a_2(s,0) + \int_0^T \sin \zeta_2 \, dT \approx a_2(s,0) + (\pi - \zeta_2(s,0))T . \tag{5.17}$$

Now

$$\zeta'_{2,s}(s,0) = C'_2 + a'_{ij}\,\zeta'_{j,T} = O(1) \ .$$

$$\zeta'_2(s,0) - \pi = O(s) \ .$$

Therefore

$$\alpha(s,0) - \frac{\pi}{2} = \zeta'_2 - \mu' = O(s) + \left\{\frac{m^2(\gamma+1)\,v_{10}v}{(2m^2+1)(\bar{\alpha}+1)v'}\right\}^{\frac{1}{2}} s^{\frac{\bar{\alpha}+1}{2}}$$

$$\approx \left\{\frac{m^2(\gamma+1)K_0v}{(2m^2+1)(\bar{\alpha}+1)v'}\right\}^{\frac{1}{2}} s^{\frac{\bar{\alpha}+1}{2}} \ . \qquad (5.18)$$

Hence,

$$\lambda_2(s,0) = \sin\alpha \approx 1$$

$$\lambda_1(s,0) = \cos\alpha \approx \left\{\frac{m^2(\gamma+1)K_0v}{(2m^2+1)(\bar{\alpha}+1)v'}\right\}^{\frac{1}{2}} s^{\frac{\bar{\alpha}+1}{2}}$$

and

$$a_2(s,0) \approx s \qquad (5.19)$$

$$a_1(s,0) \approx \frac{2}{3+\bar{\alpha}} \left\{\frac{m^2(\gamma+1)K_0v}{(2m^2+1)(\bar{\alpha}+1)v'}\right\}^{\frac{1}{2}} s^{\frac{\bar{\alpha}+3}{2}} \ . \qquad (5.20)$$

Thus, the shock is approximately of the form

$$a_1 = \frac{2}{3+\bar{\alpha}} \left\{\frac{m^2(\gamma+1)K_0v}{(2m^2+1)(\bar{\alpha}+1)v'}\right\}^{\frac{1}{2}} a_2^{\frac{\bar{\alpha}+3}{2}} = \beta a_2^{\frac{3+\bar{\alpha}}{2}} \text{ , say.} \qquad (5.21)$$

It is easy to see that, if $\bar{\alpha} = 0$, (5.21) is the same as (3.18) obtained before. Now (5.16) and (5.17) become

$$a_1(s,T) = \beta s^{\frac{\bar{\alpha}+3}{2}} - T \qquad (5.22)$$

$$a_2(s,T) = s \mp \frac{2(1-m^2)}{m} \left\{\frac{K_0v}{(\bar{\alpha}+1)(\gamma+1)(2m^2+1)v'}\right\}^{\frac{1}{2}} s^{\frac{\bar{\alpha}+1}{2}}\, T \ . \qquad (5.23)$$

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